§ 2.6 - Exact Equations

Given any ODE
\[ y' = G(x, y), \]

If we can solve it \( \exists f(x, y) \) s.t.
\[ f(x, y) = C \]
is an implicit solution. Using the chain rule,
\[ \frac{d}{dx} (f(x, y)) = \frac{df}{dx} C \]

\[ \Rightarrow f_x(x, y) + f_y(x, y) y' = 0 \]

\[ \Rightarrow y' = -\frac{f_x(x, y)}{f_y(x, y)} \]

Therefore, we must have
\[ -\frac{f_x(x, y)}{f_y(x, y)} = G(x, y), \]

If \( f(x, y) = x^2 e^y + \sin y + 5x, \)

\[ f_x(x, y) = 2x e^y + 5 \]
\[ f_y(x, y) = x^2 e^y + \cos y \]

\[ \Rightarrow y' = \frac{-2x e^y - 5}{x^2 e^y + \cos y} \]

How would we find \( f(x, y), \) starting from ?
Suppose we start with

\[ y' = G(x, y) = \frac{-M(x, y)}{N(x, y)} \]

\[ \Rightarrow \quad N(x, y) \, dy = -M(x, y) \, dx \]

\[ \Rightarrow \quad M(x, y) \, dx + N(x, y) \, dy = 0 \]

We would like \( M = f_x \) and \( N = f_y \). When this happens, the ODE is called **exact**. Recall that

\[ f_{xy} = f_{yx} \quad \text{(if they are continuous)} \]

Thus if \( M = f_x \) and \( N = f_y \), we should have

\[ M_y = N_x \]

It turns out that this is also sufficient, and so in order to check if a given ODE

\[ M \, dx + N \, dy = 0 \]

is exact, we check if

\[ M_y = N_x \]

\[ \square \quad (2xe^y + 5) \, dx + (x^2e^y + \cos y) \, dy = 0 \]

\[ \frac{\partial}{\partial y} \downarrow \quad \downarrow \frac{\partial}{\partial x} \]

\[ 2xe^y = 2xe^y \quad \checkmark \]

So this ODE is exact.
IF
\[ Mdx + Ndy = 0 \]

IS EXACT IT MEANS THAT \( \exists f \) s.t.
\[ M = f_x \text{ and } N = f_y. \]

Therefore
\[ f(x,y) = \int M(x,y) \, dx + h(y) \]

AND
\[ f(x,y) = \int N(x,y) \, dy + g(x) \]

\[ \square \]

Solve \( (2xe^y+5) \, dx + (x^2e^y + \cos y) \, dy = 0 \).

We already showed it was exact, so
\[ f(x,y) = \int 2xe^y + 5 \, dx + h(y) \]
\[ = x^2e^y + 5x + h(y) \]

AND
\[ f(x,y) = \int x^2e^y + \cos y \, dy + g(x) \]
\[ = x^2e^y + \sin y + g(x) \]

\[ \Rightarrow x^2e^y + 5x + h(y) = x^2e^y + \sin y + g(x) \]

\[ \Rightarrow h(y) - \sin y = g(x) - 5x \]

The only way this is possible is if
\[ h(y) = \sin(y) + C \quad \text{and} \quad g(x) = 5x + C \]

Plug either of these back in
\[ \Rightarrow f(x,y) = x^2e^y + \sin y + 5x = C \]

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